Today artificial neural networks are very useful to solve complex dynamic games of various types, i.e., to approximate optimal strategies with sufficient accuracy. Exemplarily four synthesis approaches for the solution of zero-sum, noncooperative dynamic games are outlined and discussed. Either value function, adjoint vector components or optimal strategies can be synthesized as functions of the state variables. In principle all approaches enable the solution of dynamic games. Nevertheless every approach has advantages and disadvantages which are discussed. The neural network training usually is very difficult and computationally very expensive. The coarse-grained parallelization FAUN 1.0-HPC-PVM of the advanced neurosimulator FAUN uses PVM subroutines and runs on heterogeneous and decentralized networks interconnecting general-purpose workstations, PCs and also high-performance computers. Computing times of days, weeks or months can be cut down to hours. An enhanced cornered rat game — formulated and analyzed in 1993 — serves as an example. Optimal strategies for cat and rat are synthesized. For this purpose open-loop representations of optimal strategies on an equidistant grid in the state space are used. An important end game modification is presented.

**Keywords**: Dynamic games; artificial neural networks; parallel computation; synthesis of optimal strategies; cornered rat game.

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1. Introduction

Consider the dynamic equations

\[ \dot{z} = f(z, u, w), \quad z(t_0) = z_0, \]

with time \( t \), initial time \( t_0 \), state \( z \in \mathbb{R}^n \) (w.l.o.g. \( z_n = t \) for explicitly time dependent systems; this is only a simplification of notation), initial state \( z_0 \), control...
\( \mathbf{u} \in \mathbb{R}^m \) for player P and control \( \mathbf{w} \in \mathbb{R}^p \) for player E. Equations (1) describe the evolution of a mathematically modeled conflict with two parties, i.e., the players P and E. The mathematical model is called a 2-person dynamic game. W.l.o.g. zero-sum, noncooperative games of the pursuit-evasion type are investigated, here. Player P is called pursuer, player E is called evader.

Exemplarily optimal strategies \( \mathbf{u}^*(\mathbf{z}) \) for P are considered. Analogously optimal strategies \( \mathbf{w}^*(\mathbf{z}) \) for E can be calculated and computed. E’s evasion strategy \( \mathbf{w}(\mathbf{z}) \) is not known to P. Thus present and future controls strategies \( \mathbf{w} \) are unknown and unpredictable for P. If P is risk averse, P may choose a worst case approach. In this case P tries to optimize the guaranteed outcome of the conflict taking into account all possible strategies \( \mathbf{w}(\mathbf{z}) \) of E.

Assume

- that \( \mathbf{u} \) is bounded by \( \mathbf{u} \in U(\mathbf{z}) \) with a closed set \( U(\mathbf{z}) \) for all \( \mathbf{z} \in \mathbb{R}^n \),
- that \( \mathbf{w} \) is bounded by \( \mathbf{w} \in W(\mathbf{z}) \) with a closed set \( W(\mathbf{z}) \) for all \( \mathbf{z} \in \mathbb{R}^n \) and
- that the conflict always terminates after finite time \( t_f - t_0 \) with

\[
\Psi_1(\mathbf{z}(t_f)) \leq \varepsilon_1 \quad \text{or} \quad \Psi_2(\mathbf{z}(t_f)) \leq \varepsilon_2 \quad \text{or} \quad \Psi_3(\mathbf{z}(t_f)) \leq \varepsilon_3 \quad \text{or \ldots} \tag{4}
\]

are called terminal sets in the state space. For some dynamic games P may have to take the responsibility for general control and state constraints \( C(\mathbf{z}, \mathbf{u}) \leq 0, C(\mathbf{z}, \mathbf{u}, \mathbf{w}) \leq 0, C(\mathbf{z}, \mathbf{w}) \leq 0 \) or \( C(\mathbf{z}) \leq 0 \) additionally. Usually these constraints can be obeyed with additional control constraints of the type (2), see B. (1996).

A strategy \( \mathbf{u}(\mathbf{z}) \) is admissible, if and only if (2) holds and \( \mathbf{u}(\mathbf{z}) \) guarantees termination after finite time, see (4), for all \( \mathbf{w} \in W(\mathbf{z}) \). All states \( \mathbf{z} \in \mathbb{R}^n \) for which an admissible strategy \( \mathbf{u}(\mathbf{z}) \) exists generate the set \( S_\mathbf{u} \) of usable states.

For a given performance index \( \Phi(\mathbf{z}(t_f)) : \mathbb{R}^n \to \mathbb{R} \) an admissible strategy \( \mathbf{u}^*(\mathbf{z}) \) is optimal, if and only if \( \mathbf{u}^*(\mathbf{z}) \) minimizes \( \Phi(\mathbf{z}(t_f)) \) for all \( \mathbf{z}_0 \in S_\mathbf{u} \) against all \( \mathbf{w} \in W(\mathbf{z}) \). For all \( \mathbf{z}_0 \in S_\mathbf{u} \) the guaranteed value \( V(\mathbf{z}_0) = \Phi(\mathbf{z}(t_f)) \) using an optimal strategy \( \mathbf{u}^*(\mathbf{z}) \) and a worst case — from the viewpoint of P — strategy \( \mathbf{w}^-(\mathbf{z}) \) of E.

A closed form solution of the dynamic game provides an optimal strategy \( \mathbf{u}^*(\mathbf{z}) \) against all \( \mathbf{w} \in W(\mathbf{z}) \). For the control process (1)–(4) \( V(\mathbf{z}) \) solves the generally nonlinear first order partial differential equation (Isaacs’ equation)

\[
\frac{\partial}{\partial \mathbf{z}} V \cdot f \left( \mathbf{z}, \mathbf{u}^* \left( \frac{\partial}{\partial \mathbf{z}} V, \mathbf{z} \right), \mathbf{w} \left( \frac{\partial}{\partial \mathbf{z}} V, \mathbf{z}, \mathbf{u}^* \left( \frac{\partial}{\partial \mathbf{z}} V, \mathbf{z} \right) \right) \right) \equiv 0 \quad \text{for all } \mathbf{z} \in S_\mathbf{u} \tag{5}
\]

with (generalized Isaacs’ minimax principle)

\[
\mathbf{u}^* \left( \frac{\partial}{\partial \mathbf{z}} V, \mathbf{z} \right) = \arg \min_{\mathbf{u} \in V(\mathbf{z})} \frac{\partial}{\partial \mathbf{z}} V \cdot f \left( \mathbf{z}, \mathbf{u}, \mathbf{w} \left( \frac{\partial}{\partial \mathbf{z}} V, \mathbf{z}, \mathbf{u} \right) \right) \quad \text{and} \tag{6}
\]

\[
\mathbf{w}^- \left( \frac{\partial}{\partial \mathbf{z}} V, \mathbf{z}, \mathbf{u} \right) = \arg \max_{\mathbf{w} \in W(\mathbf{z})} \frac{\partial}{\partial \mathbf{z}} V \cdot f \left( \mathbf{z}, \mathbf{u}, \mathbf{w} \right). \tag{7}
\]
with suitable interior conditions (singular hypermanifolds, $V$ and $\frac{\partial}{\partial z} V$ discontinuities possible)

$$\Theta_i \left( \frac{\partial}{\partial z} V^-, \frac{\partial}{\partial z} V^+, z \right) = 0 \quad \text{at} \quad \Gamma_i \left( \frac{\partial}{\partial z} V, z \right) = 0, \quad i = 1, 2, 3, \ldots, \quad (8)$$

$\Theta_i : \mathbb{R}^{3n} \to \mathbb{R}^n$, $\Gamma_i : \mathbb{R}^{2n} \to \mathbb{R}$, and suitable boundary conditions (transversality conditions)

$$\Theta_b \left( \frac{\partial}{\partial z} V, z \right) = 0 \quad \text{for} \quad z(t_f) \in S_u, \quad (9)$$

$\Theta_b : \mathbb{R}^{2n} \to \mathbb{R}^n$. Note that very difficult singular manifolds of codimension two or higher are possible — but very unusual — in the relevant parts of the state space.


From the viewpoint of P the dynamic game is solved, if at least one optimal strategy $u^*(z)$ can be computed easily for all $z \in S_u$. For real-life applications, e.g., missile, aircraft, space shuttle or robot guidance or robust optimal control, “easily” often stands for “computable with only few operations, i.e., with short, fast and reliable algorithms”. Optimal strategies $u^*(z)$ often must be real-time capable on, sometimes quite slow onboard computers, see, e.g., B. (2000), B. (1996), B. (1994), B. et al. (2001), B. et al. (2000), Järmark and Bengtsson (1989), Lachner et al. (2000), Shinar (1991) and Siemens AG et al. (1998).

2. Four Main Roads to Optimal Strategies

Four main roads lead to optimal strategies $u^*(z)$:

1. Isaacs’ equation (5) can be solved analytically for all (relevant) $z \in S_u$. All singular manifolds, i.e., discontinuities of $V(z)$ or of the gradient of $V(z)$, must be calculated explicitly. The gradient of $V(z)$ is plugged in (6) and (7). Usually this approach is applicable only for low-dimensional problems, i.e., $z \in \mathbb{R}^2$, $z \in \mathbb{R}^3$ or $z \in \mathbb{R}^4$. In addition a large class of higher dimensional Lyapunov control problems and linear-quadratic $H^\infty$ control problems can be solved, see Leitmann (1989) and Başar and Bernhard (1991) for overviews.

2. Isaacs’ equation (5) can be solved numerically for all (relevant) $z \in S_u$. Today various solution methods for nonlinear partial differential equations of first order exist, e.g., finite element methods, finite difference methods and viscosity solution approaches, see Bardi and Capuzzo-Dolcetta (1997) and Falcone and

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*aOften optimal strategies are not unique. If not unique, each optimal strategy guarantees the value $V(z)$ for P.*
The presented neural networks lead to approximated optimal strategies that are different from optimal strategies. But note that the resulting target parameter capture time $t_f$ is not influenced significantly. The result, i.e., capture or evasion of $E$, is the same with approximated optimal strategies and optimal strategies in almost all cases. Exceptions are at the barriers. A special training of additional neural networks at the barriers could improve this.

A detailed analysis of the resulting approximated strategies is possible because the optimal strategies can be calculated analytically. As the results are acceptable it is worth considering neural network strategies for more complex problems. Using neural networks always implies a basic understanding of the underlying dynamic game problem. Neural network approaches without problem analysis usually would not lead to success. Appropriately chosen neural networks can provide fast and reliable approximated optimal strategies, today.

6. Conclusions

The synthesis of optimal strategies with artificial neural networks described here started in 1993, see the diploma thesis Gabler (1993) supervised by S. Miesbach and the second author. In 1993 the classic cornered rat game, see Isaacs (1965), was
Fig. 8. Absolute error of the capture time $t_f$ obtained using the combined strategy. Note that normally the error is very low. Exceptions are the barriers, where the discontinuities in the angles provoke errors in the capture time. P is located at $(x_{P_0}, y_{P_0}) = (7, 4)$ indicated by the arrow.

enhanced. The Enhanced Cornered Rat Game is an ideal benchmark problem. Many typical difficulties, e.g., barriers and dispersal surfaces, of complex dynamic games appear. Various singular surfaces and constraints give rise to nondifferentiabilities and discontinuities of the optimal strategies. Additionally the optimal strategies for the rat E are not unique. Nevertheless optimal strategies for the cat P and the rat E can be calculated analytically, see Sec. 4. A thorough comparison to numerically computed strategies is possible.

In the diploma thesis Gabler (1993), see also Pesch et al. (1996), the optimal strategies are synthesized directly. Due to the usage of quite poor neural network training algorithms the accuracy is quite low. With the advanced neurosimulator FAUN developed by the second author since 1996, see B. (2000), B. (2000/2001) and B. et al. (2000), the accuracy achieved here is much higher, see Sec. 5. The new FAUN-HPC version, developed by the first author, was used to cut down computing times with PVM parallelization. Besides direct synthesis other synthesis
approaches with neural networks are (often more!) promising for complex dynamic game problems, see also B. (2000), B. et al. (2001) and Lachner et al. (2000). Four important approaches are outlined, compared and discussed.

In summary, artificial neural networks have the potential to revolutionize the solution of complex dynamic games. Various real life problems have been solved by the second author and his co-workers in the last decade, e.g., robust optimal space shuttle guidance, collision avoidance for cars and antiballistic missile guidance. Nevertheless the synthesis with neural networks is not — unless often asserted — a “black box” approach. Complex problems demand deep knowledge both in dynamic game theory and neuroinformatics.

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